JOURNAL OF APPROXIMATION THEORY 9, 237-254 (1973)

On the Existence of Best Rational Approximation Using a Generalized Weight Function

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Communicated by G. Meinardus

Received October 15, 1971

1. INTRODUCTION

Let X be a compact metric space; $C(X)$ the class of all continuous real valued functions defined on X; V a nonempty subset of $C(X)$; f an element of $C(X) \setminus V$; Y the set of real numbers, and $Y' = Y \cup \{-\infty\} \cup \{-\infty\}$. A function W, defined on $X \times Y$ and with range in Y', called a generalized weight function [15] if the following conditions are satisfied:

(W.1) $y_1 < y_1$ implies $W(x, y_1) \leqslant W(x, y_2)$, for all $x \in X$;

(W.2) $\operatorname{sgn} W(x, y) = \operatorname{sgn} y$, for all $x \in X$.

Other conditions on W will be required later in this paper. Let $M(g) = \sup_{x \in X} | W[x, g(x)]|$ for all $g \in C(X)$. An element $v \in V$ is called an approximation to f if $M(v - f) < \infty$. Further, v' is called a best approximation to f if it is an approximation to f and if $M(v' - f) \leqslant M(v - f)$ for every $v \in V$.

In this paper we consider the problem of the existence of a best approximation to f. The set V of approximating functions will be a nonlinear class of functions, closely related to a class of generalized rational functions defined in the next section. The case where V is a finite dimensional linear subspace of $C(X)$ was treated by Moursund [15]. Sufficient conditions for the generalized weight function W and the class of approximating functions, which guarantee the existence of a best approximation to f, are given in Section 6. Some applications are given in Section 7. Because of $W(x, y) = y$ (ordinary Chebyshev approximation) previously results needed by Gilormini and Rice can be corrected (see 7.3 below).

* This work was done while the author was an "Aspirant van het Belgisch Nationaal Fonds voor Wetenschappelijk Onderzoek" (University of Leuven, Belgium) and an "Alexander von Humboldt-Stipendiat" (University of Göttingen, Germany).

The use of a generalized weight function in approximation theory is an interesting feature to handle problems with constraints for the error curve. Some of these problems are: one-sided Chebyshev approximation and Chebyshev approximation with interpolatory constraints (see [15, p. 436], restricted range approximation (see [22]), and finding optimal starting values for the calculation of the square root (see [16]).

2. Definition of the Class of Approximating Functions

The set $C(X)$ is an associative algebra if addition, scalar multiplication and multiplication between the elements of this set, are defined in the usual way [7, p. 124]. An element $s \in C(X)$ is said to be regular for the multiplication if the relation $s \cdot g = s \cdot h$, with $g, h \in C(X)$, implies $g = h$ [28, p. 25]. It is not hard to prove the following property of the regular elements of $C(X)$.

LEMMA 1. *An element $s \in C(X)$ is regular for the multiplication if and only if $s(x) \neq 0$ for every x in a dense subset of X.*

Let P and Q be the finite-dimensional linear subspaces of $C(X)$ generated by the linear independent functions $h_1, h_2, \ldots, h_m$ and $g_1, g_2, \ldots, g_n$, respectively. The set of the regular elements of $C(X)$ in Q be denoted by Q'. Consider an element (p, q) of the product space $P \times Q'$ and let $D = \{x: x \in X \text{ and } q(x) \neq 0\}$. The real valued function r defined by $r(x) = p(x)/q(x)$ for every $x \in D$, is called a generalized rational function associated to (p, q). It is important to note that the domain D of r is a dense subset of X (see Lemma 1). The set of all generalized rational functions, associated to the elements of $P \times Q'$ is denoted by $R_{m,n}(X)$ or $R(X)$.

Some subsets of $C(X)$, which are closely related to the set $R(X)$ are defined as follows. The set $S_{m,n}(X)$ or $S(X)$ consists of the generalized rational functions which are defined everywhere in X or $S(X) = C(X) \cap R(X)$. The set $T_{m,n}(X)$ or $T(X)$ consists of the elements $t \in C(X)$ which are the continuous extension to X of a generalized rational function. This means that for every $t \in T(X)$ there exists an element $r \in R(X)$ such that $t(x) = r(x)$ for every x in the domain of r. A characterization of the generalized rational functions having a continuous extension, is given in the next lemma [8, p. 54].

LEMMA 2. *Let r be a generalized rational function with D as domain. A necessary and sufficient condition, such that r has a continuous extension t in $C(X)$ is that $\lim_{y \to x, y \in D} r(y)$ exists for every $x \in X \backslash D$.*

In some special cases it is possible to prove a criterion, which is equivalent to the one given in Lemma 2. For example if $X = [a, b]$, with a and b finite, and if $R_{m,n}(X)$ is a class of ordinary rational functions. This means that $h_i(x) = x^{i-1}$ for $i = 1, 2,..., m$ and $g_i(x) = x^{i-1}$ for $i = 1, 2,..., n$.

LEMMA 3. *Let $X = [a, b]$ and $R(X)$ be a class of ordinary rational functions. If D is the domain of $r \in R(X)$, then r has a continuous extension in $C(X)$ if and only if* $\sup_{x \in D} |r(x)| < \infty$.

For concluding this section, the definition of the image point of a generalized rational function is given. Let $r \in R(X)$ be associated with the element (p, q) of $P \times Q'$, then there exist real numbers $a_1, a_2,..., a_m$ and $b_1, b_2,..., b_n$ such that $p = \sum_{i=1}^{m} a_i h_i$ and $q = \sum_{i=1}^{n} b_i g_i$. The element $c = (a_1, a_2,..., a_m, b_1,..., b_n)$ of Y^{m+n} is called the image point of r [3, p. 178]. If t is an element of $T(X)$ such that t is the continuous extension of r, then an image point of r is also called an image point of t.

3. THE EXISTENCE PROBLEM

The norm $\| g \|$ of an element $g \in C(X)$ be defined by $\| g \| = \max_{x \in X} | g(x)|$. Let $T(X)$ be the set of the continuous extensions of the generalized rational functions of $R(X)$. Consider a function $f \in C(X) \backslash T(X)$ and a generalized weight function W. Let $F = \inf M(r - f)$ with $r \in T(X)$. We assume that $T(X)$ is not empty and that there exists at least one approximation to f in $T(X)$ or $F < \infty$. In this paper we want to find sufficient conditions for W and the base functions $h_1, h_2,..., h_m$ and $g_1, g_2,..., g_n$ such that $M(r' - f) = F$ for at least one $r' \in T(X)$. According to the definition given in the introduction, the function r' is called a best approximation to f.

The problem of the existence of a best approximation has already been considered in certain special cases. It has been studied in the case of Chebyshev approximation, using ordinary rational functions as approximating functions, e.g. by Kirchberger [13, p. 21], Walsh [23], N. I. Achieser [1, p. 53], Rice [19, p. 77], and Cheney [4, p. 154]. It has also been considered when the approximating functions are different from ordinary rational functions, e.g. by Collatz [5, p. 71; 6, p. 325], Goldstein [10, p. 431], Boehm [2, p. 23], Newman and Shapiro [17, p. 250], Meinardus [14, p. 149], Cheney [4, p. 155], Werner [24, p. 383], Dunham [9, p. 444], Rice [20, p. 77], and Stanko [21]. It has been studied in the case of Chebyshev approximation with constraints, e.g. by Gilormini [10, p. 20], and in the case of approximations using a continuous generalized weight function, e.g. by Moursund and Taylor [17, p. 888].

A special difficulty by solving existence problems in rational approximation using a generalized weight function is the following. Suppose $r \in T(X)$,

$$M(r - f) \leqslant F + 1 \qquad \text{and} \qquad r \cdot q = p \tag{3.1}$$

with $p \in P$ and $q \in Q'$. In the case of Chebyshev approximation the relation (3.1) implies the existence of a real number K, independent of r, such that

$$\| r \| \leqslant K < \infty. \tag{3.2}$$

This property is of fundamental importance in proving the possible existence of a best Chebyshev approximation to f. If the approximation problem is defined by using a generalized weight function W, then the relation (3.1) does in general not imply the relation (3.2). For an illustration of this remark we give an example.

EXAMPLE 1. Let $X = [-4, 1]$ with the usual metric and consider the generalized weight function W, defined by $W(x, y) = y$ if $x \in [-4, 0]$ and $W(x, y) = x \cdot y$ if $x \in (0, 1]$. Let $h_1(x) = x$, $g_1(x) = 1$, $g_2(x) = s(x)$ for every $x \in X$, where $s(x) = x^2$ if $x \in [0, 1]$ and $s(x) = -x$ if $x \in [-4, 0]$; $f(x) = 0$ for $x = -4, -2, 0, 1$ and $f(x) = -2$ for $x = -3, -1$ and linear between these points. Let $T = T_{1,2}(X)$, then $\inf_{r \in T} M(r - f) = 1$. Consider $r_i(x) = x/(1/i + s(x))$ for $i = 1, 2,...$ then $\lim_{i \to \infty} M(r_i - f) = 1$. For $i = 1, 2,...$ we have $M(r_i - f) \leqslant 2$ but $\lim_{i \to \infty} \| r_i \| = \infty$. This means that a number K satisfying (3.2) does not exist in this case. An analoguous example can be found in [26].

In order to solve the existence problem, the following method has been used. In the first place, it is required that W satisfy certain conditions which guarantee that an element $r \in T(X)$, satisfying (3.1), has an image point in a bounded subset of Y^{m+n}. Then the existence of a sequence $\{r_i\}$ in $T(X)$ is proved so that the corresponding sequence of image points has a limit c in Y^{m+n} and $\lim_{i \to \infty} M(r_i - f) = F$. Requiring certain conditions for the base functions $\{h_i\}$ and $\{g_i\}$ it is then possible to prove that c is the image point of an element $r' \in T(X)$. This r' will be a best approximation to f if W satisfies some further conditions. We shall now discuss every point of this method in detail.

4. Boundedness Properties for the Set of Image Points

Let $r \in T(X)$, then there exist real numbers $a_1, a_2, ..., a_m$ and $b_1, b_2, ..., b_n$ such that

$$p = \sum_{i=1}^{m} a_i h_i, \qquad q = \sum_{i=1}^{n} b_i g_i \qquad \text{and} \qquad r \cdot q = p. \tag{4.1}$$

Without loss of generality we may assume that

$$\| q \| = \max_{x \in X} | q(x) | = 1. \tag{4.2}$$

This relation implies the existence of a real number N, independent of q, such that

$$\max_{1 \leqslant i \leqslant n} | b_i | \leqslant N.$$

Suppose the existence of a real number K, such that

$$M(r - f) \leqslant K. \tag{4.3}$$

In order to get the existence of a real number L, independent of r, such that

$$\max_{1 \leqslant i \leqslant m} | a_i | \leqslant L, \tag{4.4}$$

the generalized weight function W has to satisfy certain conditions. We consider two of them here separately.

A first possibility: suppose W satisfies the condition

(W.3′) Let $g \in C(X)$. For every positive real number K there exists a real number K' such that $M(g) \leqslant K$ implies $\| g \| \leqslant K'$.

The relation (4.3) implies then the existence of a real number K', independent of r, such that $\| r - f \| \leqslant K'$ or $\| r \| \leqslant \| f \| + K'$. Because the relations (4.1) and (4.2) hold we get $\| p \| \leqslant \| r \| \cdot \| q \| = \| r \|$. Consequently, there exists a real number L such that (4.4) holds, and we obtain the following result.

THEOREM 1. *If the following conditions hold*

(a) *W satisfies* (W.3′);

(b) *$r \in T(X)$ satisfies* (4.1)–(4.3),

then the image point $(a_1, a_2, \ldots, a_m, b_1, \ldots, b_n)$ of r is an element of a bounded subset of Y^{m+n}.

A second possibility: suppose W satisfies the condition

(W.3) $\lim_{|y| \to \infty} | W(x, y) | = \infty$ for all $x \in X$.

This condition for W was also considered by Moursund in 1966 [15, p. 435] and is related to a condition used by Jackson in 1924 [12, p. 215]. The following theorem is similar to Theorem 1 and is proved using a method which is related to the one used by Moursund [15, p. 438] for the linear case.

THEOREM 2. *If the following conditions hold*

(a) *W satisfies* (W.3),

(b) $r \in T(X)$ *satisfies* (4.1)–(4.3),

then the image point $(a_1, a_2, \ldots, a_m, b_1, \ldots, b_n)$ *of* r *is an element of a bounded subset of* Y^{m+n}.

Proof. Because of the condition (4.2) we only have to prove that $(a_1, a_2, \ldots, a_m)$ is an element of a bounded subset of Y^m. Suppose that this were not the case, then there exist sequences $\{p_i\}$ in P, $\{q_i\}$ in Q, and $\{r_i\}$ in $T(X)$ such that, with

$$p_i = \sum_{j=1}^{m} a_{ij} \cdot h_j, \qquad q_i = \sum_{j=1}^{n} b_{ij} \cdot g_j, \tag{1}$$

$$A_i = (a_{i1}, a_{i2}, \ldots, a_{im}), \qquad B_i = (b_{i1}, b_{i2}, \ldots, b_{in}), \tag{2}$$

the following relations are satisfied:

$$r_i \cdot q_i = p_i; \qquad \| q_i \| = 1; \qquad M(r_i - f) \leqslant K; \tag{3}$$

$$\lim_{i\to\infty} \| A_i \| = \infty \qquad \text{with} \quad \| A_i \| = \max_{1 \leqslant j \leqslant m} | a_{ij} |. \tag{4}$$

The relation (4) implies the existence of a sequence $\{A_i\}$ and an element A' in Y^m such that $\| A_i \| \neq 0$ and

$$\lim_{i\to\infty} A_i/\| A_i \| = A' \qquad \text{with} \quad \| A' \| = 1. \tag{5}$$

Using (3) we get the existence of a sequence $\{q_i\}$ and an element q' in Q such that

$$\lim_{i\to\infty} q_i = q' \qquad \text{with} \quad \| q' \| = 1. \tag{6}$$

Let $A' = (c_1, c_2, \ldots, c_m)$; $p' = \sum_{i=1}^{m} c_i \cdot h_i$ and $q' = \sum_{i=1}^{n} d_i \cdot g_i$. Using (5) and the linear independence of $h_1, h_2, \ldots, h_m$, we get the existence of an element $x' \in X$ such that $p'(x') \neq 0$. Take

$$p_i' = \sum_{j=1}^{m} A_{ij}/\| A_i \| \cdot h_j \qquad \text{if} \quad \| A_i \| \neq 0$$

then (5) and (6) imply

$$\lim_{i\to\infty} p_i'(x') = p'(x') \qquad \text{and} \qquad \lim_{i\to\infty} q_i(x') = q'(x') \quad \text{with} \ | q'(x')| \leqslant 1. \tag{7}$$

Because $p'(x') \neq 0$ there exists a real number N' such that $p_i'(x') \neq 0$ for

$i > N'$. Put $y_i = r_i(x') - f(x')$ for $i > N'$. Using (4) and (7) it is not hard to see that $\lim_{i\to\infty} | y_i | = \infty$. Together with (W.3) we get

$$\lim_{i\to\infty} | W[x', y_i]| = \lim_{i\to\infty} | W[x', r_i(x') - f(x')]| = \infty.$$

This result contradicts the fact that $M(r_i - f) \leqslant K$. Consequently, the image point of r is an element of a bounded subset of Y^{m+n}.

Remark. Using simple examples we can show that a generalized weight function W, satisfying (W.3′) does not always satisfies (W.3). Suppose $X = [0, 1]$ and let $W(0, y) = \operatorname{sgn} y$ and $W(x, y) = y$ if $x \in (0, 1]$. Then $M(g) \leqslant K$ implies $\| g \| \leqslant K$; consequently, (W.3′) is satisfied. It is, however, clear that (W.3) does not hold for $x = 0$. It is also possible that a generalized weight function satisfies (W.3) without satisfying (W.3′). The generalized weight function used in Example 1 illustrates this fact.

In the proofs of Theorems 1 and 2 we only used the conditions (W.3) or (W.3′); consequently, these theorems remain correct, even if W does not satisfy (W.1) or (W.2).

5. Conditions for the Base Functions and the Generalized Weight Function

Using Theorems 1 and 2 we get the existence of a sequence $\{r_i\}$ in $T(X)$ such that $\lim_{i\to\infty} M(r_i - f) = F$ and such that the sequence of the image points of r_i has a limit in Y^{m+n}. This can be seen as follows. The definition of F implies the existence of sequences $\{p_i\}$ in P, $\{q_i\}$ in Q and $\{r_i\}$ in $T(X)$ such that

$$\lim_{i\to\infty} M(r_i - f) = F \quad \text{with} \quad r_i \cdot q_i = p_i \quad \text{and} \quad \| q_i \| = 1. \tag{5.1}$$

Further, the following notations are used:

$$\begin{aligned} p_i &= \sum_{i=1}^{m} a_{ij} \cdot h_j \quad \text{and} \quad A_i = (a_{i1}, a_{i2}, \ldots, a_{im}), \\ q_i &= \sum_{j=1}^{n} b_{ij} \cdot g_j \quad \text{and} \quad B_i = (b_{i1}, b_{i2}, \ldots, b_{in}). \end{aligned} \tag{5.2}$$

Using (5.1) we get the existence of a real number K, such that $M(r_i - f) \leqslant F + 1$ for $i > K$. If W satisfies (W.3′) or (W.3) then Theorems 1 or 2 implies the existence of real numbers L and N such that $\| A_i \| \leqslant L$ and $\| B_i \| \leqslant N$ for $i > K$. These relations imply that A_i and B_i belong to compact subsets of Y^m and Y^n, respectively. Consequently, we get the existence of a sequence

$\{A_i\}$ and of an element $A' = (a_1', a_2', ..., a_m')$ in Y^m such that $\lim_{i\to\infty} A_i = A'$ and $\| A' \| \leqslant L$. There exist also a sequence $\{B_i\}$ and an element $B' = (b_1', b_2', ..., b_n')$ in Y^n such that $\lim_{i\to\infty} B_i = B'$ and $\| B' \| \leqslant N$. This means that $T(X)$ contains a sequence $\{r_i\}$ satisfying (5.1), whose image points have $(a_1', a_2', ..., a_m', b_1', ..., b_n')$ as limit.

Let p' and q' be defined as follows:

$$p' = \sum_{i=1}^{m} a_i' \cdot h_i \qquad \text{and} \qquad q' = \sum_{i=1}^{n} b_i' \cdot g_i . \tag{5.3}$$

The problem of the existence of a best approximation to f is now reduced to the problem of the existence of an element r' in $T(X)$ satisfying $r' \cdot q' = p'$ and $M(r' - f) = F$. In order to solve this problem, we have to answer certain questions:

(i) Is q' regular for the multiplication in $C(X)$? This is only the case if the set $D = \{x: x \in X \text{ and } q'(x) \neq 0\}$ is dense in X (see Lemma 1).

(ii) Suppose D is dense in X and let r be the generalized rational function associated to the element (p', q'), does r then have a continuous extension r' in $C(X)$? In order to answer this question we have to verify the existence of $\lim_{y\to x, y\in D} r(y)$ for every $x \in X \backslash D$ (see Lemma 2).

(iii) If r' exists, does it satisfy $M(r' - f) = F$?

Before answering these question we give some examples illustrating the fact that the answer on every question might be negative.

EXAMPLE 2. Let $X = \{0, 1, 3\}$. Let $h_1(x) = g_1(x) = 1$ and $g_2(x) = x$ for every $x \in X$; $f(0) = 1$ and $f(1) = f(3) = 0$; $W(x, y) = y$ for every $x \in X$ and $T = T_{1,2}(X)$. Consider the functions $r_i(x) = 1/(1 + i \cdot x)$ for $i = 1, 2,...$ then $M(r_i - f) = 1/(1 + i)$; consequently, $\lim_{i\to\infty} M(r_i - f) = 0$. This implies $F = \inf_{r\in T} M(r - f) = 0$. The elements p' and q' as defined by (5.3) satisfy $p'(x) = 0$ and $q'(x) = x/3$ for every $x \in X$. The element q' does not belong to Q' because the set $\{1, 3\}$ is not dense in X. The function $r' \in C(X)$ defined by $r'(x) = 0$ for every $x \in X$, satisfies $r' \cdot q' = p'$ and is an element of T. Because $r'(0) - f(0) = 1$ we have $M(r' - f) = 1$ or r' is not a best approximation to f. It is clear that f has no best approximation in T. A similar example can be found in [2, p. 22] and [20, p. 78].

EXAMPLE 3. Let us consider the same problem as given in Example 1. The elements p' and q' as defined by (5.3) satisfy $p'(x) = x/4$ and $q'(x) = s(x)/4$ for every $x \in X$. It is clear that $q' \in Q'$. There is, however, no $r' \in T$ such that $r' \cdot q' = p'$ because the element $r \in R_{1,2}(X)$ defined by $r(x) = x/{-x}$ for $x \in [-4, 0)$ and $r(x) = x/x^2$ for $x \in (0, 1]$ has no continuous extension in $C(X)$. It can be proved [26, p. 100] that f has no best approximation in T. For a similar example see [27, p. 936–937], where the Chebyshev criterion has been used.

EXAMPLE 4. Let $f(x) = x$ for every x in $X = [0, 2]$. Let T be the class of the constant functions in $C(X)$. Consider the approximation problem defined by the following generalized weight function: $W(x, y) = y$ if $x \in (0, 2]$; $W(0, y) = y$ if $y < 1$; $W(0, y) = \infty$ if $y \geqslant 1$. We have $F = \inf_{r \in T} M(r - f) = 1$ and the elements $r_i = 1 - 1/i$ of T satisfy $\lim_{i \to \infty} M(r_i - f) = 1$. The elements p' and q' as defined by (5.3) satisfy $p'(x) = q'(x) = 1$ for every $x \in X$. The element $r'(x) = 1$ of T satisfies $r' \cdot q' = p'$ but $M(r' - f) = \infty$ because $r'(0) - f(0) = 1$. Consequently, a best approximation to f does not exist. A similar example was given in [15, p. 439].

Example 2 shows that the element q' as defined by (5.3) is not always an element of Q'. The reason, therefore, is that the set of regular elements $q \in Q'$ satisfying $\| q \| = 1$, is not always closed in $C(X)$. This set is, however, closed if the base functions $g_1, g_2, \ldots, g_n$ of Q satisfy the following condition:

(B.1) every nonzero element $q \in Q$ is different from zero in a dense subset of X.

This condition is equal to the "dense nonzero property" as used, e.g. by Boehm [2, p. 20], Newman and Shapiro [18, p. 245], Gilormini [10, p. 20], and Rice [20, p. 84].

Example 3 shows that, even if (B.1) is satisfied, there does not always exist an element $r' \in C(X)$ such that $r' \cdot q' = p'$. According to Lemma 2 this is only the case if $\lim_{y \to x, y \in D} p'(y)/q'(y)$ exist for every $x \in X \backslash D$, where $D = \{x\colon x \in X \text{ and } q'(x) \neq 0\}$. It is especially by considering this condition that our theory, if applied to the Chebyshev approximation problem, is different from the one given by Gilormini [10, p. 19–25] and Rice [20, p. 77–84]. In order to get that $r' \in T(X)$ we introduce the following conditions:

(B.2′) every generalized rational function $r \in R(X)$ satisfying $\sup_{x \in D} | r(x)| < \infty$, where D is the domain of r, has a continuous extension in $C(X)$.

(B.2) every generalized rational function $r = p/q \in R(X)$ satisfying $p(x) = 0$ for every $x \in X \backslash D$, where D is the domain of r, has a continuous extension in $C(X)$.

Example 4 shows that r' is only a best approximation to f in all cases if W satisfies a further condition, which can be stated as follows [15, p. 439]:

(W.4) for every $x \in X$ and $t \in Y$: $\lim_{y \to t, |y| < |t|} W(x, y) = W(x, t)$.

6. Sufficient Conditions for the Existence of a Best Approximation

Using the conditions for the base functions and the generalized weight function, given in the preceeding sections, we can now prove some lemmas, which give us important indications about the possible existence of a best approximation to f in $T(X)$. Afterwards two existence theorems are formulated. The following notations will be needed. If W satisfies (W.3) or (W.3′) then there exist real numbers L, N and sequences $\{p_i\}$ in P, $\{q_i\}$ in Q and $\{r_i\}$ in $T(X)$ such that

$$\lim_{i\to\infty} M(r_i - f) = F \quad \text{with} \quad r_i \cdot q_i = p_i \quad \text{and} \quad \| q_i \| = 1, \tag{6.1}$$

$$\lim_{i\to\infty} p_i = p' = \sum_{j=1}^{m} a_j' \cdot h_j \quad \text{with} \quad \max_{1\leqslant j\leqslant m} | a_j' | \leqslant L, \tag{6.2}$$

$$\lim_{i\to\infty} q_i = q' = \sum_{j=1}^{n} b_j' \cdot g_j \quad \text{with} \quad \max_{1\leqslant j\leqslant n} | b_j' | \leqslant N \quad \text{and} \quad \| q' \| = 1. \tag{6.3}$$

If $g_1, g_2, \ldots, g_n$ satisfy (B.1) then $q'(x) \neq 0$ in a dense subset D of X. Let r be the generalized rational function, defined by

$$r(x) = p'(x)/q'(x) \qquad \text{for every} \quad x \in D. \tag{6.4}$$

Lemma 4. *If the following conditions hold*

(a) *W satisfies* (W.3′),

(b) *the base functions $g_1, g_2, \ldots, g_n$ satisfy* (B.1),

(c) *there exists at least one approximation to f in $T(X)$,*

then there exists a generalized rational function $r \in R(X)$ such that $\sup_{x\in D} | r(x)| < \infty$, *where D is the domain of r.*

Proof. The conditions (a), (b), and (c) imply the existence of the sequences $\{p_i\}$, $\{q_i\}$, and $\{r_i\}$ such that (6.1), (6.2), and (6.3) hold. From (6.1) and (a) follows the existence of real numbers K and K' such that

$$\| r_i \| \leqslant K' \qquad \text{if} \quad i > K. \tag{1}$$

Let r be defined as in (6.4) then we get

$$\lim_{i\to\infty} p_i(x)/q_i(x) = r(x) \qquad \text{for every } x \in D. \tag{2}$$

The relations (6.1), (1), and (2) imply

$$\lim_{i\to\infty} | r_i(x)| = | r(x)| \leqslant K' \qquad \text{for every } x \in D.$$

This result concludes the proof of Lemma 4.

LEMMA 5. *If the following conditions hold*

(a) *W satisfies* (W.3),

(b) *the base functions* $g_1, g_2, \ldots, g_n$ *satisfy* (B.1),

(c) *there exists at least one approximation to f in* $T(X)$,

then there exists a generalized rational function $r = p/q$ *in* $R(X)$, *such that* $p(x) = 0$ *for every* $x \in X \backslash D$, *where D is the domain of r.*

Proof. Let p', q', and r be defined as in (6.2)–(6.4). Suppose that $p'(x) \neq 0$ for some $x \in X \backslash D$. From (6.2) follows $\lim_{i \to \infty} p_i(x) = p'(x) \neq 0$. Together with (6.1) this implies that $q_i(x) \neq 0$ for i sufficiently high. Consequently,

$$\lim_{i \to \infty} r_i(x) = \lim_{i \to \infty} p_i(x)/q_i(x) = p'(x) \cdot \lim_{i \to \infty} 1/q_i(x) = \infty.$$

Put $y_i = r_i(x) - f(x)$ then $\lim_{i \to \infty} | y_i | = \infty$, and using (W.3) we get

$$\lim_{i \to \infty} | W[x, r_i(x) - f(x)]| = \infty.$$

This result is in contradiction to (6.1). Consequently, it is not possible that $p'(x) \neq 0$ for some $x \in X \backslash D$.

LEMMA 6. *If the following conditions hold*

(a) *W satisfies* (W.1), (W.2), (W.3) *or* (W.3′), *and* (W.4),

(b) *the base functions* $g_1, g_2, \ldots, g_n$ *satisfy* (B.1),

(c) *there exists at least one approximation to f in* $T(X)$,

then there exists a generalized rational function $r \in R(X)$ *such that*

$$\sup_{x \in D} | W[x, r(x) - f(x)]| \leqslant F, \tag{1.5}$$

where D is the domain of r.

Proof. Consider the generalized rational function r as defined in (6.4). Suppose that the relation (6.5) is not satisfied, then there exists an element $x \in D$ and a positive number d such that

$$| W[x, r(x) - f(x)]| \geqslant F + d. \tag{1}$$

Using (W.2), (1) means that $r(x) - f(x) \neq 0$. Since $q'(x) \neq 0$ for every $x \in D$ we get, using (6.1) until (6.4),

$$\lim_{i \to \infty}[r_i(x) - f(x)] = r(x) - f(x) \neq 0. \tag{2}$$

Consequently, there exists a real number K' such that

$$\operatorname{sgn}[r_i(x) - f(x)] = \operatorname{sgn}[r(x) - f(x)] \qquad \text{for} \quad i > K'. \tag{3}$$

The relation (6.1) implies the existence of a real number K'' such that

$$M(r_i - f) \leqslant F + d/2 \qquad \text{for} \quad i > K''. \tag{4}$$

Take $K = \max\{K', K''\}$ then (1) and (4) imply

$$| W[x, r_i(x) - f(x)]| < | W[x, r(x) - f(x)]| \qquad \text{for} \quad i > K. \tag{5}$$

Using (W.4), the relations (2), (3), and (5) imply

$$\lim_{i\to\infty} | W[x, r_i(x) - f(x)]| = | W[x, r(x) - f(x)]|.$$

Combining this result with (1) we get

$$M(r_i - f) > F + d/2 \qquad \text{for } i \text{ sufficiently high.}$$

This result is in contradiction with (6.1); consequently, (6.5) must hold.

A result which is similar to the one expressed in Lemma 6 has also been proved by Newman and Shapiro [18, p. 250] for the case of Chebyshev approximation. The result given in Lemma 6 is very important and implies the existence of a best approximation in some special cases. We mention some of them in the following corollaries.

COROLLARY 1 OF LEMMA 6. *Let D_r be the domain of a generalized rational function $r \in R(X)$. If the conditions* (a), (b), *and* (c) *of Lemma* 6 *hold and $D_r = X$ for every $r \in R(X)$, then there exists a best approximation to f in $T(X)$.*

Consider the case where $n = 1$ and $g_1(x) = 1$ for all $x \in X$. Every element $r \in T(X)$ can then be written as a linear combination of $h_1, h_2, ..., h_m$ or $r = \sum_{i=1}^{m} a_i \cdot h_i$. This case corresponds to the linear case, which was treated by Moursund [15]. It is clear that condition (B.1) is satisfied and that $R(X) = S(X) = T(X) = P$.

COROLLARY 2 OF LEMMA 6. *If W satisfies* (W.1), (W.2), (W.3) *or* (W.3′), *and* (W.4) *and if there exists at least one approximation to f in P, then there exists a best approximation to f in P.*

The result, expressed in Corollary 2, has been obtained by Moursund [15, p. 439], in the case where W satisfies (W.1), (W.2), (W.3), and (W.4).

LEMMA 7. *If the following conditions hold*

(a) *W satisfies* (W.1), (W.2), (W.3′), *and* (W.4),

(b) *the base functions satisfy* (B.1) *and* (B.2′),

(c) *there exists at least one approximation to f in T(X),*

then there exists an element r′ in T(X) and a dense subset D of X such that

$$\sup_{x\in D} | W[x, r'(x) - f(x)]| \leqslant F,$$

where $D = X$ if X is the domain of every $r \in R(X)$.

Proof. Let r be the generalized rational function, as defined in (6.4). Applying Lemmas 4 and 6 we get

$$\sup_{x\in D} | W[x, r(x) - f(x)]| \leqslant F \qquad \text{and} \qquad \sup_{x\in D} | r(x)| < \infty. \tag{1}$$

Since (B.2′) is supposed to hold, there exists an element $r' \in T(X)$ such that

$$r(x) = r'(x) \qquad \text{for every} \quad x \in D. \tag{2}$$

The relations (1) and (2) obviously imply the wanted result.

LEMMA 8. *If the following conditions hold*

(a) *W satisfies* (W.1), (W.2), (W.3), *and* (W.4),

(b) *the base functions satisfy* (B.1) *and* (B.2),

(c) *there exists at least one approximation to f in T(X),*

then there exists an element r′ in T(X) and a dense subset D of X such that

$$\sup_{x\in D} | W[x, r'(x) - f(x)]| \leqslant F,$$

where $D = X$ if X is the domain of every $r \in R(X)$.

Proof. The same method as in the proof of Lemma 7 can be used, applying Lemma 5 instead of Lemma 4.

Remark. Consider the element $r \in R(X)$ as defined in (6.4). In Lemmas 4 and 5 we proved some properties of r, implying that only certain elements in $R(X)$ need to have a continuous extension in $C(X)$. It is not hard to see that if the base functions satisfy (B.2) they also satisfy (B.2′), but not the contrary. Example 1 shows that Lemma 3 is not applicable if the condition (W.3′) is replaced by (W.3). This is the reason for introducing two different conditions (B.2′) and (B.2), for the base functions.

The Existence Theorems

In [26, pp. 97–98] we gave an example to show that the Lemmas 7 and 8 in general do not imply the existence of a best approximation to f in $T(X)$. Therefore, we introduced the following condition for the generalized weight function [26, p. 93]:

(W.5) Let K be an arbitrary real number, then the relation $\sup_{x \in D} | W[x, g(x)]| \leqslant K$ implies $M(g) \leqslant K$ for every dense subset D of X and every $g \in C(X)$.

The conditions for W and the base functions, which guarantee the existence of a best approximation to f in $T(X)$ can now be formulated into two theorems.

THEOREM 3. *If the following conditions hold*

(a) *W satisfies* (W.1), (W.2), (W.3′), (W.4), *and* (W.5),

(b) *the base functions satisfy* (B.1) *and* (B.2′),

(c) *there exists at least one approximation to f in $T(X)$,*

then there exists a best approximation to f in $T(X)$.

Proof. Lemma 7 implies the existence of an element $r' \in T(X)$ such that

$$\sup_{x \in D} | W[x, r'(x) - f(x)]| \leqslant F,$$

where D is a dense subset of X. Because $r' - f$ is an element of $C(X)$ we get $M(r' - f) \leqslant F$, by using (W.5). The definition of F and the fact that $r' \in T(X)$ imply that $M(r' - f) = F$ or r' is a best approximation to f.

In the same way, by using Lemma 8 instead of Lemma 7 the following theorem can be proved.

THEOREM 4. *If the following conditions hold*

(a) *W satisfies* (W.1), (W.2), (W.3), (W.4), *and* (W.5),

(b) *the base functions satisfy* (B.1) *and* (B.2),

(c) *there exists at least one approximation to f in $T(X)$,*

then there exists a best approximation to f in $T(X)$.

7. SOME APPLICATIONS OF THE EXISTENCE THEOREMS

7.1. *First Application: The Generalized Weight Function Is Continuous*

If the generalized weight function W satisfies (W.3) and is continuous in $X \times Y$, then W satisfies also (W.3′) [17, p. 888]. Every continuous W satisfies obviously (W.4) and (W.5) and there exists an approximation to every f in $T(X)$. As a consequence of Theorem 3 we get the following result.

THEOREM 5. *If the following conditions hold*

(a) *W satisfies* (W.1), (W.2), *and* (W.3) *and is continuous in* $X \times Y$,

(b) *the base functions satisfy* (B.1) *and* (B.2′),

then there exists a best approximation to f in $T(X)$.

7.2. *Second Application: The Approximating Functions Are Ordinary Rational Functions*

If $X = [a, b]$ and $R(X)$ is a set of ordinary rational functions, then we have $S(X) = T(X)$ and the conditions (B.1) and (B.2′) (see Lemma 3) are satisfied.

THEOREM 6. *If the following conditions hold*

(a) *W satisfies* (W.1), (W.2), (W.3′), (W.4), *and* (W.5),

(b) $R(X)$ *is a set of ordinary rational functions and* $X = [a, b]$,

(c) *there exists at least one approximation to f in* $S(X)$,

then there exists a best approximation to f in $S(X)$.

This theorem follows directly from Theorem 3. A similar result where the condition (W.3′) is replaced by (W.3) is not always correct. An example of this phenomenon is given in [26, p. 100]. Since the sets $R_{2,2}(X)$ and $R_{1,n}(X)$ with $n \geqslant 1$ of ordinary rational functions satisfy the conditions (B.1) and (B.2), we obtain the following result as a corollary of Theorem 4.

THEOREM 7. *If the following conditions hold*

(a) *W satisfies* (W.1), (W.2), (W.3), (W.4), *and* (W.5),

(b) $R_{m,n}(X)$ *is a set of ordinary rational functions, with* $m = n = 2$ *or* $m = 1$, $n \geqslant 1$ *and* $X = [a, b]$,

(c) *there exists at least one approximation to f in* $S(X)$,

then there exists a best approximation to f in $S(X)$.

If the generalized weight function is continuous and the approximating functions are ordinary rational functions, then Theorem 5 implies the following result.

THEOREM 8. *If the following conditions hold*

(a) *W satisfies* (W.1), (W.2), *and* (W.3) *and is continuous in* $X \times Y$,

(b) $R_{m,n}(X)$ *is a set of ordinary rational functions and* $X = [a, b]$,

then there exists a best approximation to f in $S(X)$.

We note that the same result has been obtained by Moursund and Taylor in [17, p. 888].

7.3. *Third Application: Ordinary Chebyshev Approximation*

Consider the generalized weight function W, defined by $W(x, y) = y$ for every $x \in X$. This W obviously satisfies (W.1), (W.2), (W.3), (W.3′), (W.4), and (W.5). As a corollary of Theorem 3 we get the following result.

THEOREM 9. *If the base functions satisfy* (B.1) *and* (B.2′) *then there exists a best Chebyshev approximation to f in $T(X)$.*

Related results have been given by Gilormini [10, p. 20] and Rice [20, p. 84]. However, the condition (B.2′) was not considered by them. As we showed in [27] Gilormini's result is, therefore, wrong. A similar remark can be made about Rice's result.

7.4. *Fourth Application: One-Sided Chebyshev Approximation*

Let $W(x, y) = y$ if $y \geqslant 0$ and $W(x, y) = -\infty$ if $y < 0$. This generalized weight function defines an approximation problem where $\| r - f \|$ has to be minimized, such that $r(x) - f(x) \geqslant 0$ for all $x \in X$. It is not hard to see that W satisfies the conditions (W.1), (W.2), (W.3′), (W.3), (W.4), and (W.5). Consequently, Theorem 3 implies the following result.

THEOREM 10. *If the following conditions hold*:

(a) *the base functions satisfy* (B.1) *and* (B.2′),

(b) *there exists at least one $r \in T(X)$ such that $r(x) - f(x) \geqslant 0$,*

then there exists a best approximation to f in $T(X)$.

7.5. *Fifth Application: Chebyshev Approximation with Interpolatory Constraints*

Let $X' = \{x_1, x_2, ..., x_s\}$ be a finite subset of X. Let $W(x, y) = y$ if $x \notin X'$; $W(x_i, y) = \infty$ if $y > 0$, $W(x_i, y) = 0$ if $y = 0$, $W(x_i, y) = -\infty$ if $y < 0$ for $i = 1, 2, ..., s$. In the corresponding approximation problem $\| r - f \|$ is minimized such that $r(x_i) = f(x_i)$ for $i = 1, 2, ..., s$. The generalized weight function W satisfies (W.1), (W.2), (W.3′), (W.3), and (W.4) but not (W.5).

This means that Theorems 3 and 4 cannot be applied. In [25, p. 96] an example is given to show that a best approximation to f might not exist, even if the interpolatory conditions are satisfied by some $r \in T(X)$.

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